§5 Integral dependence and valuations

$A \subseteq B$ subring

§5.1 integral dependence

An element $x \in B$ is said to be integral over A, if x is a root of a monic polynomial with coeficient in A.

i.e. $x^n + a_1 x^{n-1} + \dots + a_n = 0$, $a_i \in A$.

Example : $x = \frac{r}{s} \in \mathbb{Q}$ integral over $\mathbb{Z}$ iff $x \in \mathbb{Z}$.

Prop 5.1. TFAE.

i). $x \in B$ integral over A

ii). $A[x]$ = f.g. A-mod.

iii). $\exists$ subring C s.t. $A[x] \subseteq C \subseteq B$ and C = f.g. as an A-mod.

iv). $\exists$ faithful $A[x]$-module M, which is f.g. as an A-module

Pf : i) $\Rightarrow$ ii) $x^n + a_1 x^{n-1} + \dots + a_n = 0 \Rightarrow x^{n+r} = -(a_1 x^{n+r-1} + \dots + a_n x^r)$

ii) $\Rightarrow$ iii) $C := A[x]$

iii) $\Rightarrow$ iv) $M := C$ $(yM = 0 \Rightarrow y \cdot 1 = 0 \Rightarrow y = 0)$

iv) $\Rightarrow$ i) $\phi : M \longrightarrow M$

$m \mapsto xm$

$\overset{(2.4)}{\Rightarrow} \exists a_i$ s.t. $\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0$

$\Rightarrow (x^n + a_1 x^{n-1} + \cdots + a_n) \cdot m = 0 \quad \forall m \in M$

$\overset{\text{faithful}}{\Longrightarrow} \quad x^n + a_1 x^{n-1} + \cdots + a_n = 0$ □

Cor 5.2 $x_1, \cdots, x_n \in B$ integral over A. Then

$A[x_1, \cdots, x_n] =$ f.g. A-module

Pf: $A_i := A[x_1, \cdots, x_i]$

$A_i =$ f.g. A_{i-1}-module

(2.16) $\Rightarrow A_n =$ f.g. as A-module.

Cor 5.3. $C := \{x \in B \mid x \text{ integral over } A\} \subset B$ is a subring

Pf: $x, y \in C \Rightarrow A[x,y] = \text{f.g. } A\text{-mod} \overset{(5.1\,\text{iii})}{\Rightarrow} x \pm y,\ xy \in C$ □

Def C as in (5.3).

i) C is called integral closure of A in B.

ii) A is called integrally closed in B, if $C = A$.

iii) B is called integral over A, if $C = B$

iv) $f : A \to B$ ring hom. f is called integral or B is called an integral A-alg., if $B/f(A) = $ integral.

Cor 5.3 $\Leftrightarrow$ $\boxed{\text{f.t.} + \text{integral} = \text{finite}}$

Cor : 5.4. (transitivity of integral dependence) $A \subseteq B \subseteq C$

C/B & $B/A = $ int. $\Rightarrow$ $C/A = $ int.

Pf: $\forall\, x \in C \overset{C/B}{\Rightarrow} \exists\ x^n + b_1 x^{n-1} + \cdots + b_n = 0$

$\overset{B/A}{\Rightarrow} B' := A[b_1, \cdots, b_n]$ f.g. A-mod.

$\Rightarrow x \in B'[x] = A[b_1, \cdots, b_n, x] \overset{(2.16)}{=}$ f.g. A-mod

Cor 5.5 : $C :=$ integral closure of A in B. Then

C is integral closed in B. i.e. $\overline{\overline{A}} = \overline{A}$ in B.

Pf: $x \in B$ int. over $C \Rightarrow x$ int. over $A \Rightarrow x \in C$ $\square$

Prop 5.6 : $B = A$-integral $\Rightarrow \begin{cases} B/\mathfrak{b} = A/\mathfrak{b}^c\text{-int} \\ S^{-1}B = S^{-1}A\text{-int}. \end{cases}$

Pf: $x^n + a_1 x^{n-1} + \dots + a_n = 0$

$$\Rightarrow \begin{cases} \overline{x}^n + \overline{a}_1 \overline{x}^{n-1} + \dots + \overline{a}_n = 0 \\ \left(\frac{x}{s}\right)^n + \frac{a_1}{s} \cdot \left(\frac{x}{s}\right)^{n-1} + \dots + \frac{a_n}{s^n} = 0 \end{cases}$$ $\square$

§ 5.2 the going-up theorem

Prop 5.7 $A \subseteq B$ integral domains, B/A = integral. Then

$$B = \text{field} \Leftrightarrow A = \text{field}.$$

Pf: $\Rightarrow)$ $\forall x \in A \setminus \{0\}$

$$\Rightarrow (x^{-1})^n + a_1 (x^{-1})^{n-1} + \cdots + a_n = 0$$

$$\Rightarrow x^{-1} = -(a_1 + a_2 x + \cdots + a_n x^{n-1}) \in A$$

$\Leftarrow)$ $\forall y \in B \setminus \{0\}$.

$y^m + a_1' y^{m-1} + \cdots + a_m' = 0$ $\quad a_i' \in A$

(minimal degree)

$$\Rightarrow a_m' \neq 0 \quad (\text{or, } y^{m-1} + a_1' y^{m-2} + \cdots + a_{m-1}' = 0)$$

$$\Rightarrow y(y^{m-1} + a_1' y^{m-2} + \cdots + a_{m-1}') + a_m' = 0$$

$$\Rightarrow y^{-1} = -a_m'^{-1}(y^{m-1} + a_1' y^{m-2} + \cdots + a_{m-1}') \quad \square$$

Cor 5.8 $A \subseteq B$, B/A = integral. $\mathfrak{p} = \mathfrak{q}^c$.

$$\begin{tikzcd}
\mathfrak{q} = \text{maximal} \arrow[rr, Leftrightarrow] \arrow[d, Leftrightarrow] & & \mathfrak{p} = \text{maximal} \arrow[d, Leftrightarrow] \\
B/\mathfrak{q} = \text{field} \arrow[rr, Leftrightarrow] & & A/\mathfrak{p} = \text{field} \\
& B/\mathfrak{q} \text{ integral over } A/\mathfrak{p} \arrow[u, Rightarrow] & \\
& B \text{ integral over } A \arrow[u, Rightarrow] &
\end{tikzcd}$$

Pf

Cor 5.9. B/A = integral. $\mathfrak{q}, \mathfrak{q}' \in \operatorname{Spec} B$

$$\mathfrak{q} \subseteq \mathfrak{q}' \ \& \ \mathfrak{p} := \mathfrak{q}^c = \mathfrak{q}'^c \Rightarrow \mathfrak{q} = \mathfrak{q}'.$$

Pf: $S := A - \mathfrak{q}^c \Rightarrow S^{-1}B / A_{\mathfrak{p}}$ = integral

(*) $S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{q}'$

(**) $(S^{-1}\mathfrak{q})^c = S^{-1}\mathfrak{p} = (S^{-1}\mathfrak{q}')^c$

$$S^{-1}\mathfrak{p} = \text{maximal} \overset{(**)}{\Rightarrow} S^{-1}\mathfrak{q}, S^{-1}\mathfrak{q}' = \text{maximal}$$

$$\overset{(*)}{\Rightarrow} S^{-1}\mathfrak{q} = S^{-1}\mathfrak{q}'$$

$$\overset{(3.11\text{ iv})}{\Rightarrow} \mathfrak{q} = \mathfrak{q}'. \qquad \square$$

Thm 5.10 $B/A = \text{integral} \Rightarrow \operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$.
$\mathfrak{q} \mapsto i^{-1}(\mathfrak{q})$

Pf:
$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "\beta"] \\
A_{\mathfrak{p}} \arrow[r, "i_{\mathfrak{p}}"'] & B_{\mathfrak{p}} \\
\mathfrak{n}^c & \mathfrak{n}
\end{tikzcd}$$

integral
$\Downarrow \cdot (5.6)$
integral

$\forall\, \mathfrak{n} \triangleleft B_{\mathfrak{p}}$ maximal $\Rightarrow \mathfrak{m} := \mathfrak{n} \cap A_{\mathfrak{p}} \triangleleft A_{\mathfrak{p}}$ maximal

$\Rightarrow \alpha^{-1}(\mathfrak{m}) = \mathfrak{p}$

$\mathfrak{q} := \beta^{-1}(\mathfrak{n}) \in \operatorname{Spec} B$.

$$i^{-1}(\mathfrak{q}) = (\beta \circ i)^{-1}(\mathfrak{n}) = (i_{\mathfrak{p}} \circ \alpha)^{-1}(\mathfrak{n}) = \mathfrak{p}. \qquad \square$$

Thm 5.11 (Going-up theorem)

$$\begin{tikzcd}
B & \mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_m & \overset{\exists}{\subseteq \cdots \subseteq \mathfrak{q}_n} \\
A \arrow[u, "\text{integral}"'] & \mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_m \arrow[u, dotted, dash] & \subseteq \cdots \subseteq \mathfrak{p}_n \arrow[u, dotted, dash]
\end{tikzcd}$$

Pf: induction $\Rightarrow$ reduce to $m=1$, $n=2$. & $\mathfrak{p}_1 \neq \mathfrak{p}_2$

$A/\mathfrak{p}_1 \to B/\mathfrak{q}_1$ integral $\Rightarrow \exists\, \bar{\mathfrak{q}}_2$ s.t. $\bar{\mathfrak{q}}_2^{\,c} = \bar{\mathfrak{p}}_2 \Rightarrow \mathfrak{q}_2 \supseteq \mathfrak{q}_1$ & $\mathfrak{q}_2^{\,c} = \mathfrak{p}_2$.

§ 5.3 integrally closed integral domains.

(The going-down theorem)

Prop. 5.12 C = integral closure of A in B. $S \subset A$. m.c.subset.

$$\Rightarrow \quad S^{-1}C = \text{integral closure of } S^{-1}A \text{ in } S^{-1}B.$$

Pf: $(5.6) \Rightarrow S^{-1}C / S^{-1}A$ = integral.

$\forall \ \frac{b}{s} \in S^{-1}B$ with $\left(\frac{b}{s}\right)^n + \left(\frac{a_1}{s_1}\right)\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s_n} = 0$

$$\xRightarrow{t := s_1 \cdots s_n} (bt)^n + a_1 s \cdot s_2 \cdots s_n \cdot (bt)^{n-1} + \cdots + a_n s^n \cdot s_1^n \cdots s_{n-1}^n s_n^{n-1} = 0$$

$$\Rightarrow bt \in C \quad \Rightarrow \frac{b}{s} = \frac{bt}{st} \in S^{-1}C \qquad \square$$

Def: A integral domain is called <u>integrally closed</u>, if it is integrally closed in its field of fractions.

UFD $\Rightarrow$ integrally closed.

e.g. $\mathbb{Z}$, $k[x_1, \cdots, x_n]$

Prop 5.13. Integral closed is a local property. i.e. TFAE

i) A = integral closed

ii) $A_{\mathfrak{p}}$ = integral closed $\forall \mathfrak{p}$ prime

iii) $A_{\mathfrak{m}}$ = integral closed $\forall \mathfrak{m}$ maximal

Pf:

$$\begin{tikzcd}
A = \text{integral clsed} \arrow[r, Leftrightarrow] & A \to C \ \text{surj} \arrow[d, Leftrightarrow] \\
A_{\mathfrak{p}} = \text{integral closed} \arrow[r, Leftrightarrow, "5.12"] & A_{\mathfrak{p}} \to C_{\mathfrak{p}} \ \text{surj} \arrow[d, Leftrightarrow] \\
A_{\mathfrak{m}} = \text{integral closed} \arrow[r, Leftrightarrow] & A_{\mathfrak{m}} \to C_{\mathfrak{m}} \ \text{surj}
\end{tikzcd}$$

Def $\mathfrak{a} \triangleleft A \subseteq B$.

i) $x \in B$ is integral over $\mathfrak{a}$, if $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some, $a_1, \cdots, a_n \in \mathfrak{a}$.

ii). integral closure of $\mathfrak{a}$ in B $:= \{ x \in B \mid \text{integral over } \mathfrak{a} \}$

Lem 5.14. $\mathfrak{a} \triangleleft A \subseteq B$ C = integral closure of A in B

$\underline{\text{integral closure of } \mathfrak{a} \text{ in } B} = \sqrt{\mathfrak{a}C}$

Pf: $\forall x \in \text{LHS} \Rightarrow \begin{cases} x \in C \\ x^n + a_1 x^{n-1} + \cdots + a_n = 0 \end{cases}$

$$\Rightarrow x^n = -(a_1 x^{n-1} + \cdots + a_1) \in \mathfrak{a}C$$

$$\Rightarrow x \in \sqrt{\mathfrak{a}C}$$

$\forall x \in \sqrt{\mathfrak{a}C}$, $\Rightarrow x^n = \sum_i a_i x_i$

$$\Rightarrow x^n M \subseteq \mathfrak{a} M \quad \left(\begin{array}{l} M := A[x_1, \ldots, x_n] \\ \text{f.g. as } A\text{-mod} \end{array} \right)$$

$$\overset{(2.4)}{\Rightarrow} x^n \text{ integral over } \mathfrak{a}$$

$$\Rightarrow x \text{ integral over } \mathfrak{a}, \qquad \square$$

Prop 5.15 · $A \subseteq B$ integral domains.

· A = integrally closed

· Let $x \in B$ integral over $\mathfrak{a}$ with minimal poly.

$t^n + a_1 t^{n-1} + \cdots + a_n$ over $K = \operatorname{Frac} A$. Then

$$a_1, a_2, \ldots, a_n \in \sqrt{\mathfrak{a}}.$$

Pf: $t^n + a_1 t^{n-1} + \cdots + a_n = (t - x_1) \cdots (t - x_n)$ with $x_i \in \overline{K}$

$\boxed{x_1 = x}$

$$\bar{\mathfrak{a}} := \{ x \in \overline{K} \mid x \text{ integral over } \mathfrak{a} \}$$

$\Rightarrow \quad x_i \in \bar{\mathfrak{a}}$

A = integral closed.

$\Rightarrow \quad a_i \in \bar{\mathfrak{a}} \cap K \subseteq A$

$\overset{(5.14)}{\Rightarrow} \quad a_i \in \sqrt{\mathfrak{a}}$

Thm 5.16 (Going-down theorem)

B : integral domain

$\updownarrow$: integral

A : integral domain
& integral closed

$$\begin{tikzcd}
\mathfrak{q}_1 \supseteq \mathfrak{q}_2 \supseteq \cdots \supseteq \mathfrak{q}_m \arrow[d, dash, dotted] & \supseteq \overset{\exists}{\mathfrak{q}_{m+1}} \supseteq \cdots \supseteq \mathfrak{q}_n \arrow[d, dash, dotted] \\
\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \cdots \supseteq \mathfrak{p}_m & \supseteq \mathfrak{p}_{m+1} \supseteq \cdots \supseteq \mathfrak{p}_n
\end{tikzcd}$$

Pf: reduce ~~to~~ $m=1$ & $n=2$. $\mathfrak{p}_1 \supsetneq \mathfrak{p}_2$. $\mathfrak{q}_1^c = \mathfrak{p}_1$

WONTS: $\mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A = \mathfrak{p}_2$.

($\overset{3.16}{\Rightarrow}$ $\mathfrak{p}_2$ is contraction of a prime ideal in $B_{\mathfrak{q}_1}$)

$\forall \quad x = \frac{y}{s} \in \mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A$, $y \in \mathfrak{p}_2 B$ & $s \in B - \mathfrak{p}_1$

(5.14) $\Rightarrow$ $y \in \overline{\mathfrak{p}_2}$ $\Rightarrow$ $y^r + u_1 y^{r-1} + \cdots + u_r = 0$, $u_r \in \mathfrak{p}_2$
(minimal over K)

$$\xRightarrow{s = yx^{-1}} \quad s^r + \frac{u_1}{x} s^{r-1} + \cdots + \frac{u_r}{x^r} = 0$$

(minimal over K)

s = integral over A $\overset{5.15}{\Rightarrow}$ $\frac{u_i}{x^i} \in A$

- Suppose $x \notin \mathfrak{p}_2$. $\Rightarrow \frac{u_i}{x^i} \in \mathfrak{p}_2 \Rightarrow s^r \in \mathfrak{p}_2 B \subseteq \mathfrak{p}_1 B \subseteq \mathfrak{q}_1$ ↯

$\Rightarrow x \in \mathfrak{p}_2 \Rightarrow \mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A = \mathfrak{p}_2$. $\square$.

Prop 5.17 A = integrally closed domain

K = Frac A

L = f. sep. alg. ext. of K

B = integral closure of A in L.

$\Rightarrow \exists$ basis $v_1 \cdots v_n$ of L/K s.t.

$$B \subseteq \sum_{j=1}^{n} A v_j$$

Pf: $\forall v \in L \Rightarrow \exists\ a_0 v^r + a_1 v^{r-1} + \cdots + a_n = 0 \quad a_i \in A$.

$\Rightarrow a_0 v \in B$

$\Rightarrow$ find a basis $u_1 \cdots u_n \in B$ of L/K.

L/K = sep. $\Rightarrow L \times L \to K$ non-degenerate.

$(x, y) \mapsto T(xy)$

$\Rightarrow$ dual basis $v_1 \cdots v_n$ of L/K.

with $T(u_i v_j) = \delta_{ij}$.

$\forall\, x = \sum_i x_i v_i \in B$

$x u_i \in B \Rightarrow x_i = Tr(x u_i) \in A$

$\Rightarrow B \subseteq \sum_i A v_i$

§ 5.4. Valuation rings.

$B =$ integral domain, $K = \mathrm{Frac}\, B$

Def : B is called a valuation ring of K if $\forall x \in K^\times$, either $x \in B$ or $x^{-1} \in B$.

Why valuation? $\Gamma := K^\times / B^\times =$ abelian gp.

$$\gamma = [x] \geq \gamma' = [y] \overset{def}{\Longleftrightarrow} x/y \in B$$

$v : K^\times \longrightarrow (\Gamma, \geq)$ is an valuation.

$$\begin{cases} v(a+b) \geq \min(v(a), v(b)) \\ v(ab) = v(a) + v(b) \end{cases}$$

Prop 5.18 i) $B =$ local

ii) $B \subseteq B' \subseteq K \Rightarrow B' =$ valuation ring

iii) B is integral closed in K.

Pf : i): $m := \{ x \in B \mid x^{-1} \notin B \} = B \setminus B^\times$.

WONTS : m is an ideal.

1° $a \in B,\ x \in m \Rightarrow ax \notin B^\times \Rightarrow ax \in m$

2° $\forall x, y \in m \Rightarrow xy^{-1} \in B$ or $yx^{-1} \in B$ (assume $xy^{-1} \in B$)

$$\Rightarrow x + y = (1 + xy^{-1})\, y \in Bm \subset m$$

ii) clear

iii) $x \in K$ integral over B.

$$x^n + b_1 x^{n-1} + \cdots + b_n = 0$$

Suppose $x \notin B \Rightarrow x^{-1} \in B$

$$\Rightarrow x = -\left(b_1 + b_2 x^{-1} + \cdots + b_n x^{1-n}\right) \in B \;\lightning$$

$\forall\, K$ = field, $\forall\, \Omega$ = algebraically closed field.

$$\Sigma = \Sigma(K, \Omega) := \left\{ (A, f) \;\middle|\; \begin{array}{l} A = \text{subring of } K \\ f : A \to \Omega \text{ ring hom.} \end{array} \right\}$$

$$(A, f) \leq (A', f') \overset{\text{def}}{\Longleftrightarrow} \begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & \Omega \arrow[d, equal] \\ A' \arrow[r, "f'"] & \Omega \end{tikzcd}.$$

$\Rightarrow$ partial ordered set : $(\Sigma, \leq)$

$\forall$ chain $(A_i, f_i)_{i \in I}$ in Σ,

$$A_\infty := \bigcup_{i \in I} A_i \quad \& \quad f_\infty(a) := f_i(a) \;\; \forall a \in A_i.$$

$\Rightarrow$ (A_∞, f_∞) is an upper bound of $(A_i, f_i)_{i \in I}$ in $(\Sigma, \leq)$.

Zorn's lemma $\overset{\text{assume } \Sigma \neq \phi}{\Rightarrow}$ $\exists$ maximal element in Σ.

Lemma 5.19: Let (B, g) be a maximal element in Σ.

Then B is local with maximal ideal $\mathfrak{m} = \ker g$.

Pf: $\operatorname{Im} g \subseteq \Omega$ is integral domain. $\Rightarrow$ $\mathfrak{m}$ = prime

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \Omega \arrow[d] \\ B_{\mathfrak{m}} \arrow[r, "g_{\mathfrak{m}}"] & \Omega \end{tikzcd}$$

(B, g) = maximal $\Rightarrow$ $B = B_{\mathfrak{m}}$

$\Rightarrow$ $(B, \mathfrak{m})$ = local.

Lemma 5.20 $x \in K^{\times}$. Then $1 \notin \mathfrak{m}[x] \cap \mathfrak{m}[x^{-1}]$.

$$\mathfrak{m}[x] := \left\{ \sum_{i=0}^{n} u_i x^i \in K \;\middle|\; u_i \in \mathfrak{m} \right\} \triangleleft B[x]$$

$$\mathfrak{m}[x^{-1}] := \left\{ \sum_{i=0}^{n} v_i x^{-1} \in K \;\middle|\; v_i \in \mathfrak{m}. \right\} \triangleleft B[x^{-1}]$$

Pf: $1 = u_0 + u_1 x + \cdots + u_k x^k$ (k minimal at here)

$1 = v_0 + v_1 x^{-1} + \cdots + v_\ell x^{-\ell}$ (ℓ minimal at here)

WMA: $k \geq \ell$.

$$(1-v_0)\,x = v_1 + \dots + v_\ell\, x^{1-\ell}$$

$$v_0 \in \mathfrak{m} \Rightarrow 1 - v_0 \in B^\times \Rightarrow x^\ell = w_1 x^{\ell-1} + \dots + w_\ell$$

$$\Rightarrow 1 = u_0 + u_1 x + \dots + u_{k-1} x^{k-1} + u_k x^{k-\ell}\,(w_1 x^{\ell-1} + \dots + w_\ell).\ \lightning.$$

Theorem 5.21: (B,g) = maximal in Σ $\Rightarrow$ B = valuation ring of K.

Pf: $\forall\, x \in K^\times,$ $\overset{5.20}{\Longrightarrow}$ assume $1 \notin \mathfrak{m}[x] \lhd B[x] =: B'$

$$\Rightarrow \exists\, \mathfrak{m}[x] \subseteq \mathfrak{m}' \subsetneq B'$$

$$(\text{clear } \mathfrak{m}' \cap B = \mathfrak{m})$$

$$\Rightarrow k := B/\mathfrak{m} \hookrightarrow B'/\mathfrak{m}' =: k' = k[\bar{x}]$$

$$k'/k = \text{f.ext.}$$

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & k \arrow[d] & \subset \Omega \arrow[d, equal] \\
B[x] \arrow[r] & k' & \subset \Omega
\end{tikzcd}$$

(B,g) = maximal $\Rightarrow$ $B = B[x]$.

$\Rightarrow$ $x \in B$.

Cor 5.22 $A < K$ subring. $\bar{A}$ = integral closure of A in K

$$\bar{A} = \bigcap_{\substack{A \subseteq B \subseteq K \\ B : \text{valuation rings}}} B$$

Pf: "$\subseteq$": $B = \bar{B} \Rightarrow \bar{A} \subseteq \bar{B} = B \Rightarrow \bar{A} \subseteq \cap B$.

"$\supseteq$": $\forall x \notin \bar{A} \Rightarrow x \notin A' := A[x^{-1}]$

$\Rightarrow x^{-1}$ not unit in A'

$\Rightarrow x^{-1} \in m' \triangleleft A'$

$\Rightarrow A \hookrightarrow A' \twoheadrightarrow A'/m' =: k \subseteq \Omega := \bar{k}$

$\Rightarrow \exists (B, g)$ maximal

$$\begin{tikzcd} A' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \Omega \arrow[d, equal] \\ B \arrow[r] & \Omega \\ K \arrow[u, phantom, "\cap" rotate=-90] & \end{tikzcd}$$

$\Rightarrow x \notin B$ (or $1 = xx^{-1} \mapsto 0$ ↯)

Prop 5.23 $A \subseteq B$ integral domains. B/A = f.g.

$\forall v \in B \setminus \{0\}$, $\exists u \in A \setminus \{0\}$ s.t.

$$\begin{tikzcd} A \arrow[r, "\forall f"] \arrow[d] & \Omega & \text{with } f(u) \neq 0 \\ B \arrow[r, "\exists g"] & \Omega \arrow[u, equal] & \text{with } g(v) \neq 0 \end{tikzcd}$$

Pf: We may assume $B = A[x]$

1° x transcendental over A.

assume $v = a_0 x^n + a_1 x^{n-1} + \dots + a_n \Rightarrow u := a_0$

$\forall f$ with $f(u) \neq 0$. (i.e. $f(a_0) \neq 0$)

$\Rightarrow \exists\, \xi \in \Omega$ s.t.

$$f(a_0)\xi^n + \dots + f(a_n) \neq 0 \in \Omega$$

$$\begin{tikzcd}
A \arrow[r, "\forall f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Omega \arrow[d, equal] \\
B \arrow[r, "x \mapsto \xi", "g"'] & \Omega
\end{tikzcd}$$

$\Rightarrow \quad g(v) \neq 0$.

2°. x is algebraic over A. ($\Rightarrow$ so is v^{-1})

$a_0 x^m + a_1 x^{m-1} + \dots + a_m = 0 \qquad a_i \in A$

$b_0 v^{-n} + b_1 v^{1-n} + \dots + b_n = 0 \qquad b_i \in B$

$u := a_0 b_0$

$\forall f : A \to \Omega$ with $f(u) \neq 0$

$\left(\text{i.e. } f(a_0) \neq 0 \neq f(b_0) \right)$

$$\Rightarrow \begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Omega \arrow[d, equal] \\ A[u^{-1}] \arrow[r, "f_1"'] \arrow[d] & \Omega \arrow[d, equal] \\ C \arrow[r, "h"] & \Omega \end{tikzcd}$$

$f_1(u^{-1}) = f(u)^{-1}$

$(5.21 \Rightarrow \exists\, (C, h))$

x integral over $A[u^{-1}] \Rightarrow x \in \overline{A[u^{-1}]} \subseteq C$

$\Rightarrow B \subseteq C$

similarly $v^{-1} \in C \Rightarrow v \in C^\times \Rightarrow h(v) \neq 0$

$g := h|_B$.

Cor 5.24 $B =$ f.g. k-alg. Then

$B =$ field $\Leftrightarrow$ $B/k =$ finite alg. ext.

Pf: $A = k$, $v = 1$, $\Omega = \bar{k}$.